Chapter 6. Control Charts for Variables

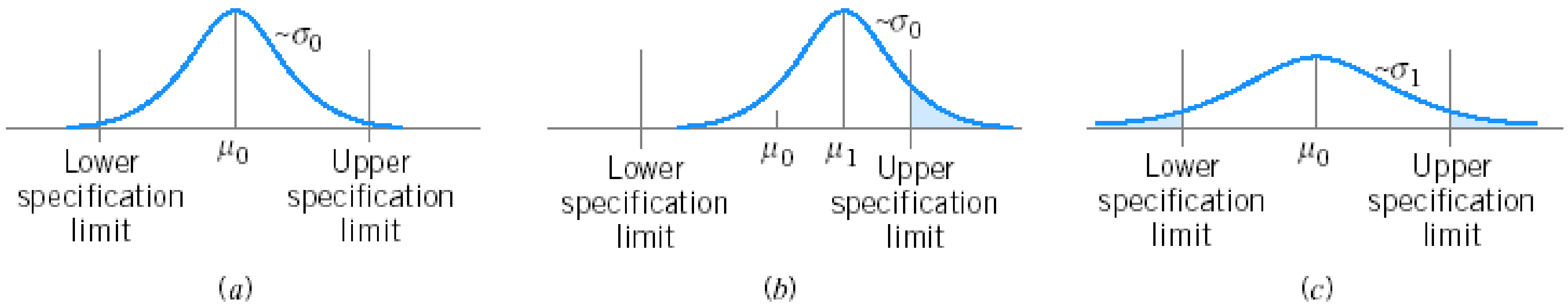


Figure 5-1 The need for controlling both process mean and process variability. (*a*) Mean and standard deviation at nominal levels. (*b*) Process mean $\mu_1 > \mu_0$. (*c*) Process standard deviation $\sigma_1 > \sigma_0$.

Control Charts for $\bar{x}$ and R

x: quantity of interest $\qquad x \sim N(\mu, \sigma)$

$x_1, x_2, \ldots, x_n$: samples of x

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

$$\bar{x} \sim N\left(\mu, \frac{\sigma}{\sqrt{n}}\right)$$

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \qquad (5\text{-}1)$$

Subgroup Data with Unknown μ and σ

$\overline{\overline{x}}$: grand average of x, best estimate for μ

$$\overline{\overline{x}} = \frac{\overline{x}_1 + \overline{x}_2 + \cdots + \overline{x}_m}{m} \qquad (5\text{-}2)$$

$$R = x_{\max} - x_{\min}$$

$R_1, R_2 \ldots, R_m$: ranges of m samples

$$\overline{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \qquad (5\text{-}3)$$

Control Limits for the $\bar{x}$ Chart

$$\text{UCL} = \bar{\bar{x}} + A_2\bar{R}$$
$$\text{Center line} = \bar{\bar{x}} \tag{5-4}$$
$$\text{LCL} = \bar{\bar{x}} - A_2\bar{R}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Limits for the R Chart

$$\text{UCL} = D_4\bar{R}$$
$$\text{Center line} = \bar{R} \tag{5-5}$$
$$\text{LCL} = D_3\bar{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

Phase I Application of $\overline{x}$ and *R* Charts

- Equations 5-4 and 5-5 are trial control limits.
  - Determined from *m* initial samples.
    - Typically 20-25 subgroups of size *n* between 3 and 5.
  - Any out-of-control points should be examined for assignable causes.
    - If assignable causes are found, discard points from calculations and revise the trial control limits.
    - Continue examination until all points plot in control.
    - Adopt resulting trial control limits for use.
    - If no assignable cause is found, there are two options.
      1. Eliminate point as if an assignable cause were found and revise limits.
      2. Retain point and consider limits appropriate for control.
  - If there are many out-of-control points they should be examined for patterns that may identify underlying process problems.

Example 5-1

Table 5-1 Flow Width Measurements (microns) for the Hard-Bake Process

Sample Number	Wafers 1	2	3	4	5	$\bar{x}_i$	R_i
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma\bar{x}_i$ = 37.6400	ΣR_i = 8.1302
						$\bar{\bar{x}}$ = 1.5056	$\bar{R}$ = 0.32521

$$\overline{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

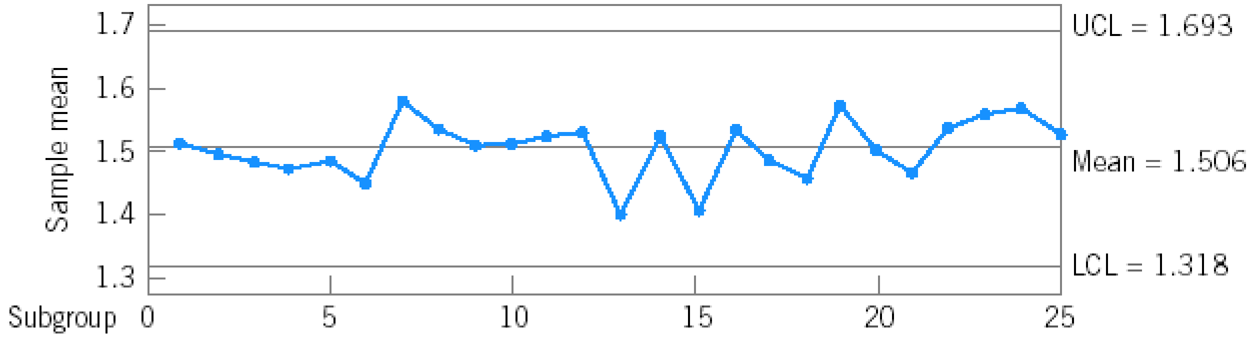
$$\text{LCL} = \overline{R}D_3 = 0.32521(0) = 0$$

$$\text{UCL} = \overline{R}D_4 = 0.32521(2.114) = 0.68749$$

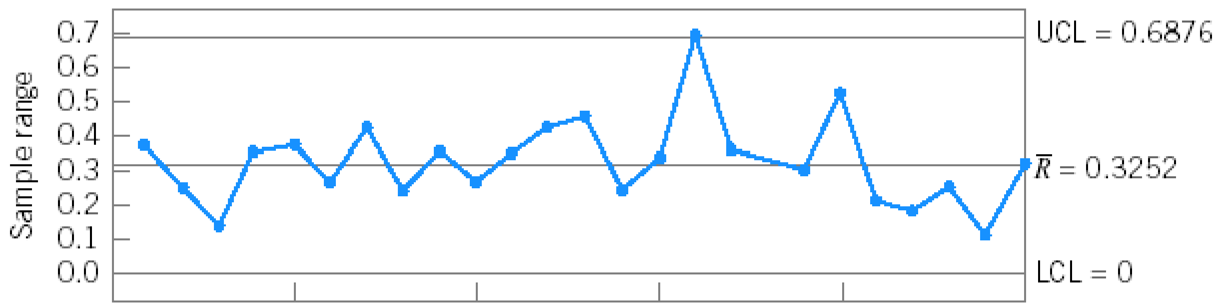
$$\overline{\overline{x}} = \frac{\sum_{i=1}^{25} \overline{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$

$$\text{UCL} = \overline{\overline{x}} + A_2\overline{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$

$$\text{LCL} = \overline{\overline{x}} - A_2\overline{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$



(*a*)



(*b*)

Figure 5-2 $\bar{x}$ and R charts (from Minitab) for flow width in the hard-bake process.

$$\hat{\sigma} = \frac{\overline{R}}{d_2} = \frac{0.32521}{2.326} = 0.1398 \text{ microns}$$

Assume spec tolerance is 1.5 +/- 0.5 micron.
Nonconformance probability:

$$\begin{aligned} p &= P\{x < 1.00\} + P\{x > 2.00\} \\ &= \Phi\left(\frac{1.00 - 1.5056}{0.1398}\right) + 1 - \Phi\left(\frac{2.00 - 1.5056}{0.1398}\right) \\ &= \Phi(-3.61660) + 1 - \Phi(3.53648) \\ &\simeq 0.00015 + 1 - 0.99980 \\ &\simeq 0.00035 \end{aligned}$$

C_p: Process Capability Ratio (PCR)

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{5-11}$$

Note: 6σ spread is the basic definition of process capability. 3σ above mean and 3σ below.

If σ is unknown, we can use $\hat{\sigma} = \frac{\overline{\text{R}}}{\text{d}_2}$. $\hat{\sigma}$ in the example is 0.1398.

$$\hat{C}_p = \frac{2.00 - 1.00}{6(0.1398)} = \frac{1.00}{0.8388} = 1.192$$

$$P = \left(\frac{1}{C_p}\right)100\%$$

P : % of specification band the process uses up. P can be estimated as:

$$\hat{P} = \left(\frac{1}{\hat{C}_p}\right)100\% = \left(\frac{1}{1.192}\right)100\% = 83.89$$

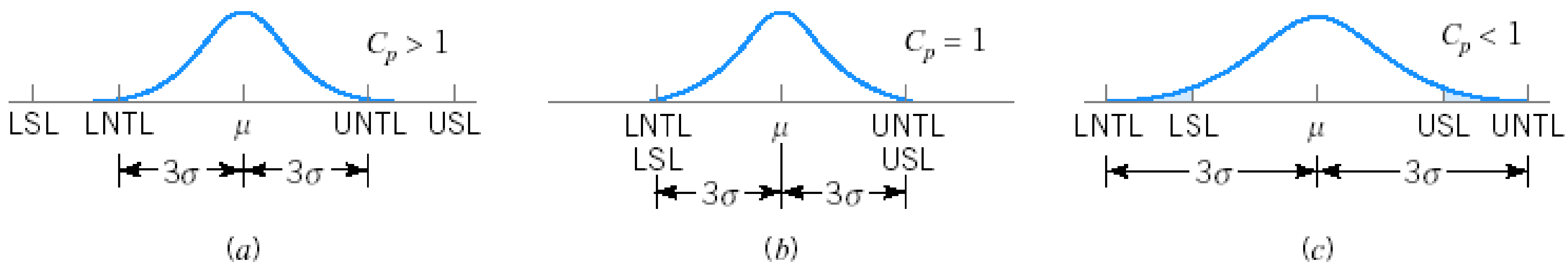


Figure 5-3 Process fallout and the process capability ratio C_p.

Revision of Control Limits and Center Lines

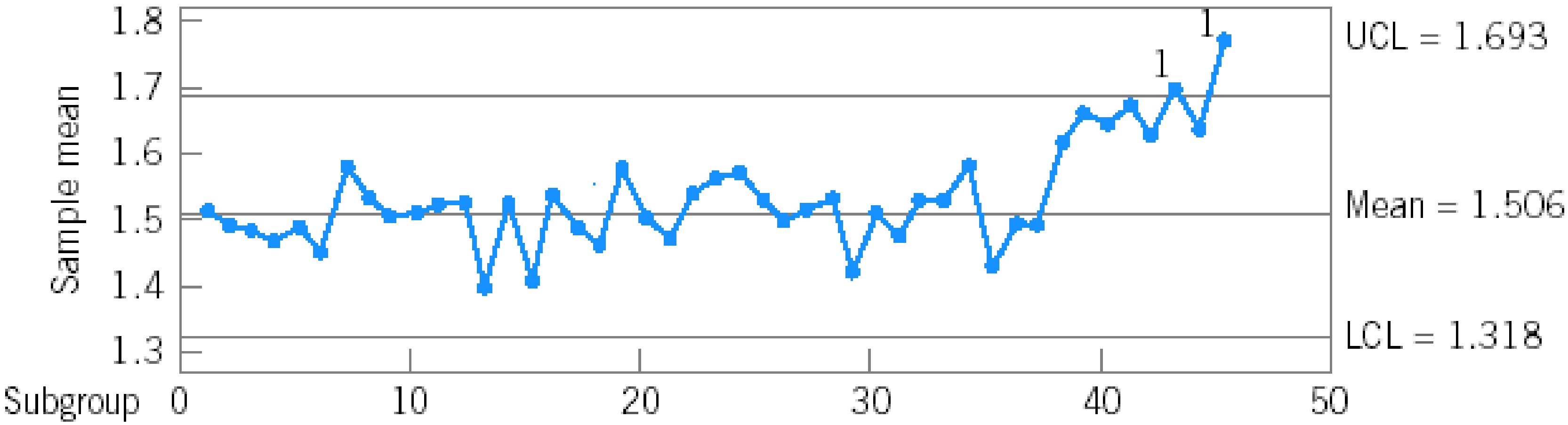
- Effective use of control charts requires periodic review and revision of control limits and center lines.
- Sometimes users replace the center line on the $\overline{X}$ chart with a target value.
- When *R* chart is out of control, out-of-control points are often eliminated to re-compute a revised value of $\overline{R}$ which is used to determine new limits and center line on *R* chart and new limits on $\overline{x}$ chart.

Phase II Operation of Charts

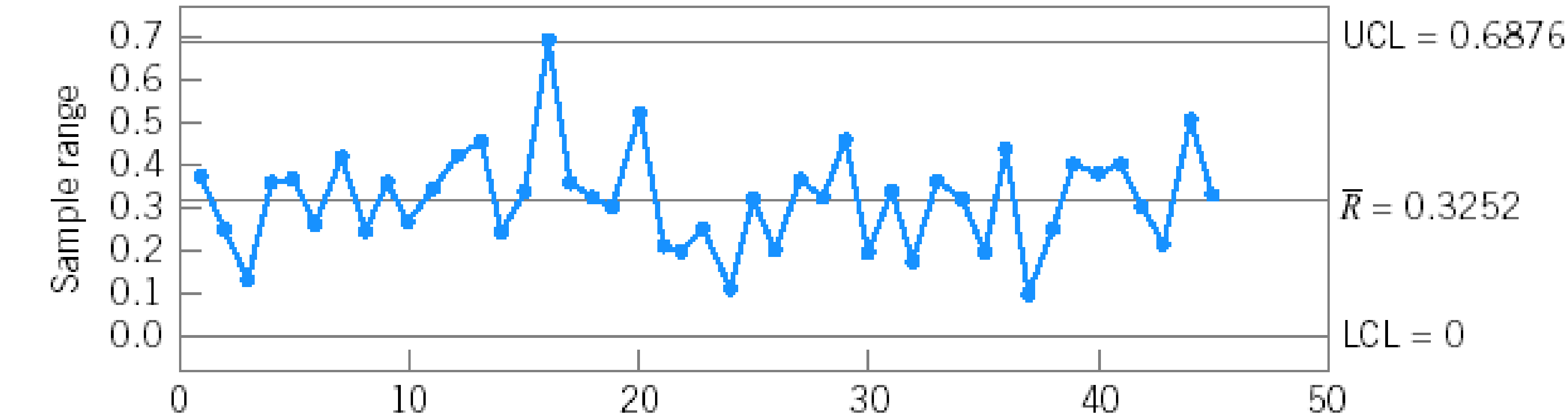
- Use of control chart for monitoring future production, after a set of reliable limits are established, is called *phase II* of control chart usage (Figure 5-4).
- A run chart showing individuals observations in each sample, called a tolerance chart or tier diagram (Figure 5-5), may reveal patterns or unusual observations in the data.

Table 5-2 Additional Samples for Example 5-1

Sample Number	Wafers						
	1	2	3	4	5	$\bar{x}_i$	R_i
26	1.4483	1.5458	1.4538	1.4303	1.6206	1.4998	0.1903
27	1.5435	1.6899	1.5830	1.3358	1.4187	1.5142	0.3541
28	1.5175	1.3446	1.4723	1.6657	1.6661	1.5332	0.3215
29	1.5454	1.0931	1.4072	1.5039	1.5264	1.4152	0.4523
30	1.4418	1.5059	1.5124	1.4620	1.6263	1.5097	0.1845
31	1.4301	1.2725	1.5945	1.5397	1.5252	1.4724	0.3220
32	1.4981	1.4506	1.6174	1.5837	1.4962	1.5292	0.1668
33	1.3009	1.5060	1.6231	1.5831	1.6454	1.5317	0.3445
34	1.4132	1.4603	1.5808	1.7111	1.7313	1.5793	0.3181
35	1.3817	1.3135	1.4953	1.4894	1.4596	1.4279	0.1818
36	1.5765	1.7014	1.4026	1.2773	1.4541	1.4824	0.4241
37	1.4936	1.4373	1.5139	1.4808	1.5293	1.4910	0.0920
38	1.5729	1.6738	1.5048	1.5651	1.7473	1.6128	0.2425
39	1.8089	1.5513	1.8250	1.4389	1.6558	1.6560	0.3861
40	1.6236	1.5393	1.6738	1.8698	1.5036	1.6420	0.3662
41	1.4120	1.7931	1.7345	1.6391	1.7791	1.6716	0.3811
42	1.7372	1.5663	1.4910	1.7809	1.5504	1.6252	0.2899
43	1.5971	1.7394	1.6832	1.6677	1.7974	1.6970	0.2003
44	1.4295	1.6536	1.9134	1.7272	1.4370	1.6321	0.4839
45	1.6217	1.8220	1.7915	1.6744	1.9404	1.7700	0.3187



(*a*)



(*b*)

Figure 5-4 Continuation of the $\bar{x}$ and R charts in Example 5-1.

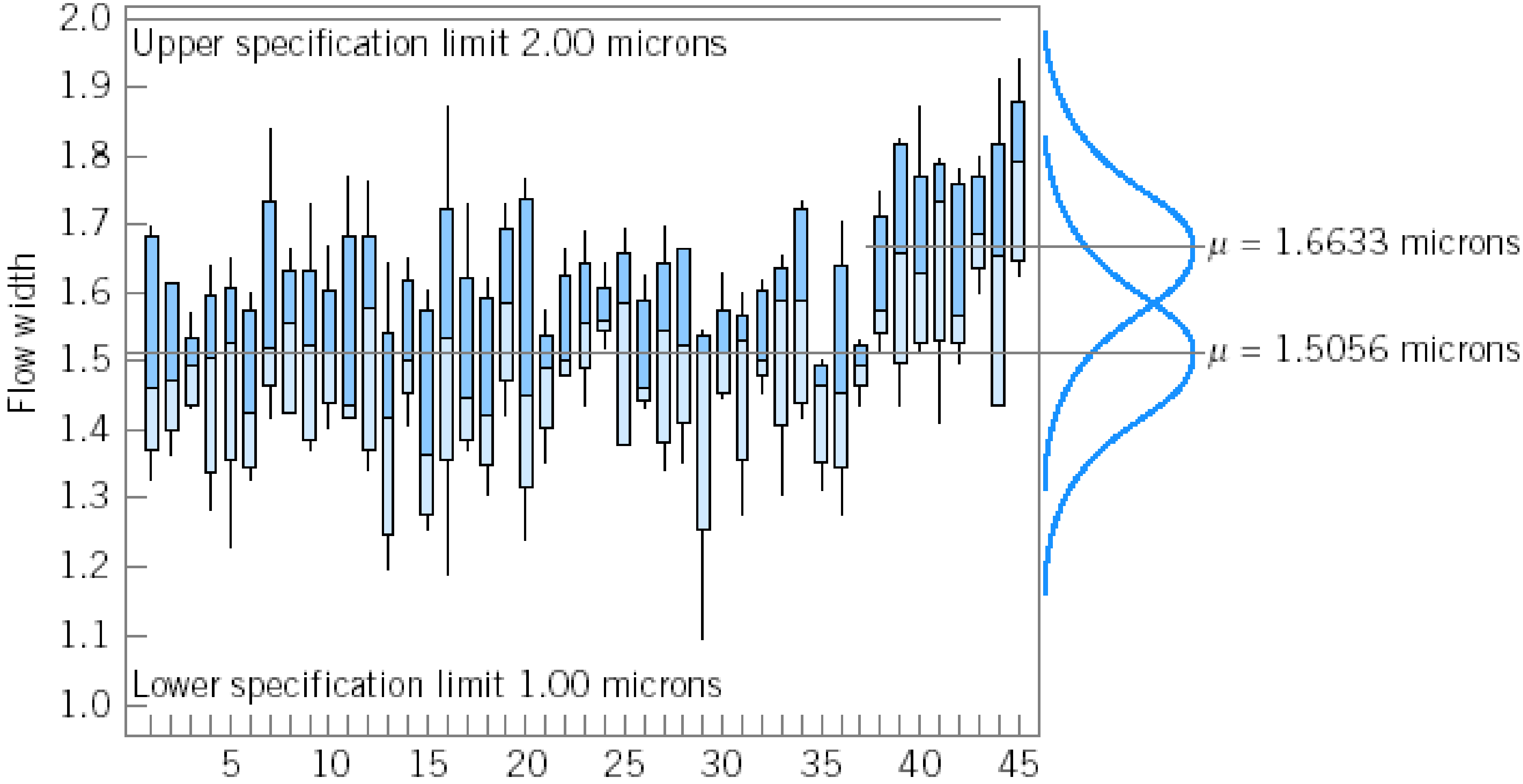


Figure 5-5 Tier chart constructed using the Minitab box plot procedure for the flow width data.

Control vs. Specification Limits

- Control limits are derived from natural process variability, or the natural tolerance limits of a process.
- Specification limits are determined externally, for example by customers or designers.
- There is no mathematical or statistical relationship between the control limits and the specification limits.

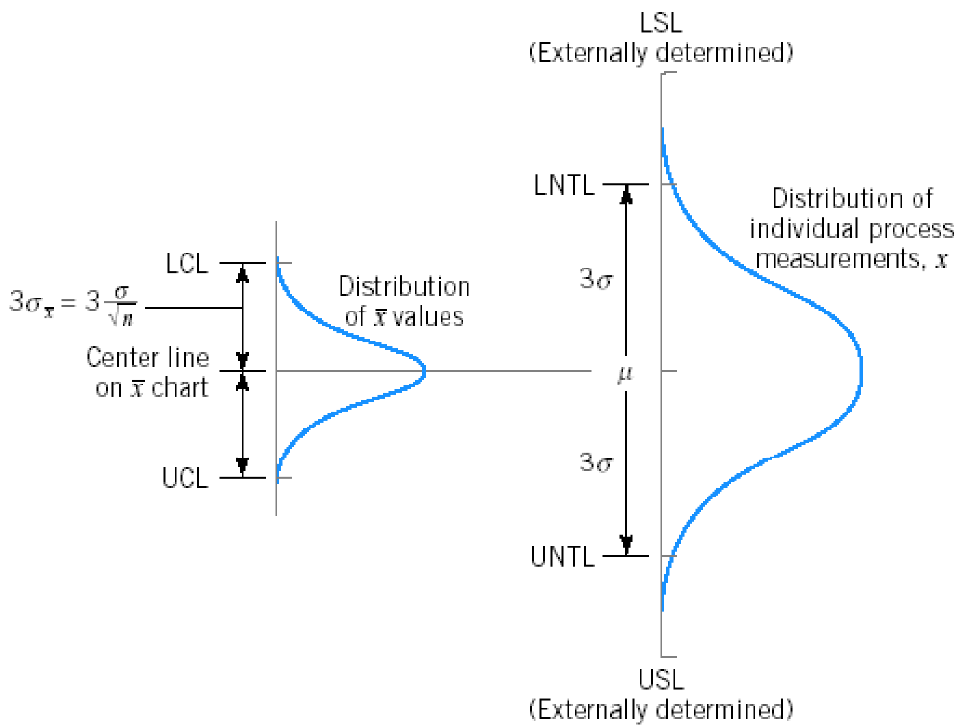


Figure 5-6 Relationship of natural tolerance limits, control limits, and specification limits.

Rational Subgroups

- $\bar{x}$ charts monitor between-sample variability.
- R charts measure within-sample variability.
- Standard deviation estimate of σ used to construct control limits is calculated from within-sample variability.

- It is not correct to estimate σ using $$s = \sqrt{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_{ij} - \bar{\bar{x}}\right)^2}{mn-1}}$$

Guidelines for Control Chart Design

- Control chart design requires specification of sample size, control limit width, and sampling frequency.
  - Exact solution requires detailed information on statistical characteristics as well as economic factors.
  - The problem of choosing sample size and sampling frequency is one of allocating sampling effort.
- For $\overline{x}$ chart, choose as small a sample size consistent with magnitude of process shift one is trying to detect. For moderate to large shifts, relatively small samples are effective. For small shifts, larger samples are needed.
- For small samples, *R* chart is relatively insensitive to changes in process standard deviation. For larger samples ($n > 10$ or 12), *s* or s^2 charts are better choices.

- NOTE: Skip Section on Changing Sample Size (pages 209-212)

Charts Based on Standard Values

$$\text{UCL} = \mu + 3\frac{\sigma}{\sqrt{n}}$$
$$\text{Center line} = \mu \qquad (5\text{-}14)$$
$$\text{LCL} = \mu - 3\frac{\sigma}{\sqrt{n}}$$

$$\text{UCL} = \mu + A\sigma$$
$$\text{Center line} = \mu \qquad (5\text{-}15)$$
$$\text{LCL} = \mu - A\sigma$$

$$\text{UCL} = D_2\sigma$$
$$\text{Center line} = d_2\sigma \qquad (5\text{-}17)$$
$$\text{LCL} = D_1\sigma$$

$D_1 = d_2 - 3d_3$

$D_2 = d_2 + 3d_3$

d_2 : mean of distribution of relative range

d_3 : standard deviation of distribution of relative range

Interpretation of $\overline{x}$ and R Charts

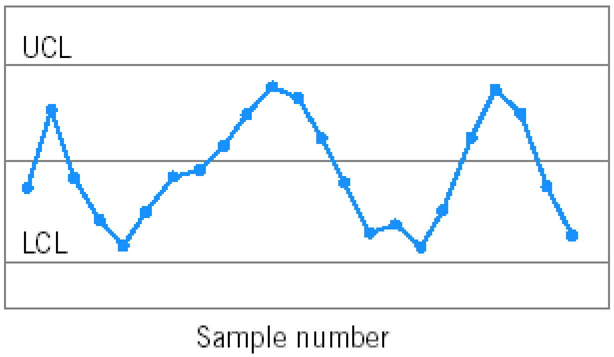
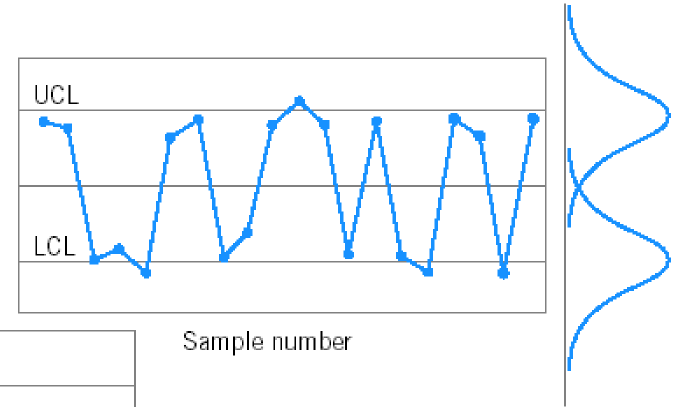


Figure 5-8 Cycles on a control chart.



A mixture pattern.

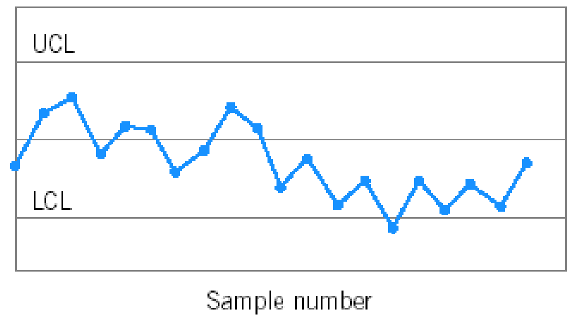


Figure 5-10 A shift in process level.

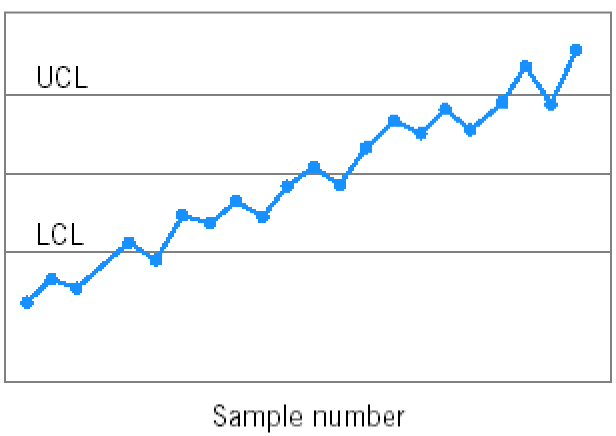


Figure 5-11 A trend in process level.

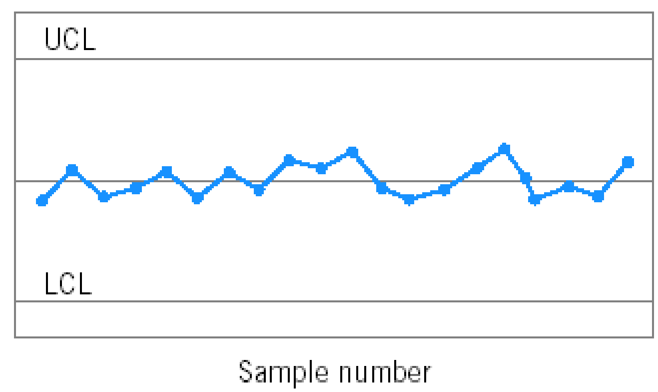


Figure 5-12 A stratification pattern.